

Radiation Reaction and the Electromagnetic Energy-Momentum of Moving Relativistic Charged Membranes

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Abstract

The charged membrane of Dirac provides a stable electron model with finite self energy. Its total mass m has been previously calculated from the Hamiltonian of the membrane. To complete the picture we evaluate it here on the basis of the energy-momentum of its self field (radiation reaction) and obtain the same result showing the consistency of the model. We show explicitly that the old 4/3-problem does not arise. The electron's stability (the vanishing of total $T^{\mu\nu}{}_{,\nu}$) is due to surface tension κ of the membrane, but the model is as simple as the point particle, with two parameters κ and e ; the surface tension parameter κ can be expressed in terms of the mass m .

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1 Introduction

In the previous papers [1, 2] we have formulated a general theory of moving relativistic charged membranes. A particularly interesting application is to the Dirac shell model of the electron [3] for which we have found, by using a more straightforward procedure, the same mass as Dirac. In that approach we assumed a spherically symmetric, radially oscillating membrane, and then calculated the canonical momentum (which contains the minimal coupling with membrane's own electromagnetic potential) and the corresponding rest mass. In the present paper we first study the self interaction of the moving membrane and the correction to the observed membrane's mass due to the radiation reaction force. So we obtain the observed (or renormalized) mass. Then we calculate, starting from the expression for the stress-energy tensor, the 4-momentum of the electromagnetic field around a moving membrane to which we add also the 4-momentum due to membrane's stress. We obtain the same expression for membrane's mass as before and also as in the previous paper. Thus we verify the consistency of various approaches.

Then we restate and formulate for the case of our spherical moving charged membrane an observation exposed in refs.[4, 5] concerning the 4-momentum (and mass) of the electromagnetic field around a moving object. We can perform this calculation by integrating the stress-energy tensor over the space-like hypersurface which is oriented along the world line of the observer with respect to whom the membrane is moving. The result is the well known factor of $3/4$ for the ratio between the energy of the electromagnetic field and its inertial mass [6]. This result has been much discussed [6] and the the researchers of the electromagnetic origin of mass came to the conclusion that not all of the electron mass is of electromagnetic origin. It has been observed [4, 5] that the integration has to be done rather over the hypersurface (time slice) which is oriented along the moving particle worldline. Then one obtains the correct expression for the 4-momentum which is consistent with the Lorentz transformations.

2 Equations of motion for membrane's centre of mass

In the following we are going to investigate the motion of membrane's centre of mass. We start from the action of an electrically charged relativistic membrane of any dimension:

$$I[x^\mu(\xi), A_\mu] = \int d^d\xi (\kappa\sqrt{|f|} + e^a\partial_a X^\mu A_\mu)\delta^D(x - X(\xi))d^Dx + \frac{1}{16\pi} \int F_{\mu\nu}F^{\mu\nu}\sqrt{|g|}d^d x \quad (1)$$

Here d is the world sheet dimension and D the space-time dimension; ξ^a , $a = 0, 1, 2, \dots, d-1$ worldsheet coordinates (parameters) and $X^\mu(\xi)$, $\mu = 0, 1, 2, \dots, D-1$ the embedding functions, $f_{ab} \equiv \partial_a X^\mu \partial_b X_\mu$ the induced metric, $f \equiv \det f_{ab}$ and e^a the electric charge current density on the world sheet. An action which is equivalent to (1) has been written down in eq.(2) of our previous publication [1].

By varying (1) with respect to X^μ we obtain the membrane's equations of motion

$$\kappa\partial_a(\sqrt{|f|}\partial^a X^\mu) + e^a\partial_a X^\nu F_\nu{}^\mu = 0 \quad (2)$$

and variation of A_μ gives the Maxwell equations

$$F^{\mu\nu}{}_{,\nu} = -4\pi j^\mu \quad , \quad j^\mu(x) = \int d^d\xi e^a\partial_a X^\mu \delta^D(x - X(\xi)) \quad (3)$$

Let us integrate eq.(2) over the world sheet element $d^d\xi$ which we can write as $d^d\xi = d\tau d\sigma$, $d\sigma = d\sigma_a n^a$, where $d\sigma_a$ is an element of a space-like hypersurface and n^a a normal vector to it. We obtain

$$\kappa \oint d\sigma_a \sqrt{|f|} \partial^a X^\mu + \int d\tau d\sigma e^a \partial_a X^\nu F_\nu{}^\mu(x) = 0 \quad (4)$$

We identify the total kinetic momentum

$$P_m^\mu \equiv \kappa \int d\sigma_a \sqrt{|f|} \partial^a X^\mu \quad (5)$$

with the centre of mass momentum. We can define the velocity \dot{X}_C^μ of the centroid world line as

$$P_m^\mu = \frac{m\dot{X}_C^\mu}{\sqrt{\dot{X}_C^2}} \quad (6)$$

where $P_m^\mu P_{m\mu} = m^2$ is the membrane's rest mass. By assuming, as usual, that only the space-like hypersurfaces Σ_1 and Σ_2 do contribute to the left integral of eq.(4), and then taking Σ_1 and Σ_2 to be infinitesimally close to each other we obtain from (4)

$$\frac{dP_m^\mu}{d\tau} + \int d\sigma e^a \partial_a X^\nu F_\nu{}^\mu(x) = 0 \quad (7)$$

This is the equation of motion for membrane's centre of mass. The electromagnetic field $F_\nu{}^\mu$ in eq.(5) can be taken to consist of the fixed external field $F_\nu{}^{\mu(\text{ext})}$ and the self-field generated by our membrane:

$$F_{\mu\nu} = F_{\mu\nu}^{(\text{ext})} + F_{\mu\nu}^{(\text{self})} \quad (8)$$

The external field can be expanded around the centroid world line X_C^μ :

$$F_{\mu\nu}^{(\text{ext})}(X) = F_{\mu\nu}^{(\text{ext})}(X_C) + \partial_\sigma F_{\mu\nu}^{(\text{ext})}|_{X=X_C} (X - X_C)^\sigma + \dots \quad (9)$$

The expression $e^a \partial_a X^\nu$ is equal to $e\dot{X}^\nu$, where $e \equiv e^a n_a$ (see [1]), and can be written as $e\dot{X}_C^\nu + e(\dot{X} - \dot{X}_C)^\nu$. By using (8-9) the equation of motion becomes

$$\frac{dP_m^\mu}{d\tau} + q\dot{X}_C^\nu F_\nu{}^{\mu(\text{ext})} + \text{higher multipoles} + \int d\sigma e\dot{X}^\nu F_\nu{}^{\mu(\text{self})} = 0 \quad (10)$$

where $q = \int e d\sigma$ is the total charge of the membrane. Eq.(10) indicates that the membrane's centre of mass moves approximately as a point particle in a fixed external field; since membrane has actually an extension, there are also contributions due to higher multipoles interaction and the self interaction.

In the special case, when there is no external field and the acceleration of membrane's centre of mass is zero, all the internal forces cancel out, so that the contribution of self

force is zero. But if we take non-zero acceleration, then, because of retardation, the internal forces do not cancel out. It is such a contribution of self interaction

$$K_{(\text{self})}^\mu = \int d\sigma e(\sigma) \dot{X}_\nu(\xi) F^{\mu\nu}(x) \quad (11)$$

that we are going to calculate now.

3 Membrane's radiation reaction and the renormalization of mass

The electromagnetic potential around our membrane can be expressed by the solution of eqs.(3)

$$A^\mu(x) = \int e^b(\xi') \delta((x - X(\xi'))^2) \partial_b X^\mu(\xi') d\xi' \quad (12)$$

satisfying the Lorenz gauge condition $\partial_\mu A^\mu = 0$. From eq.(4) we derive the following expression for the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$:

$$\begin{aligned} F^{\mu\nu}(X(\xi)) &= \int d\sigma e(\sigma') \left(\frac{1}{R^2} (X(\xi) - X(\xi'))^{[\mu} \ddot{X}(\xi')^{\nu]} - \frac{q}{R^3} (X(\xi) - X(\xi'))^{[\mu} \dot{X}(\xi')^{\nu]} \right. \\ &\quad \left. + \frac{1}{R^3} (X(\xi) - X(\xi'))^{[\mu} \dot{X}^{\nu]}(\xi') \right) \end{aligned} \quad (13)$$

$$R \equiv (X(\xi) - X(\xi'))^\sigma \dot{X}(\xi')_\sigma, \quad Q \equiv (X(\xi) - X(\xi'))^\sigma \ddot{X}(\xi')_\sigma \quad (14)$$

In the expression (13) $\xi' \equiv (\tau', \sigma'^i)$, $i = 1, 2, \dots, d-1$ refers to the source point, and therefore τ' is retarded with respect to τ of the point at which $F^{\mu\nu}$ is calculated. We now use the expansion

$$X^\mu(\xi') = X^\mu(\tau + u, \sigma') = X^\mu(\tau) + u \dot{X}^\mu + \frac{1}{2} u^2 \ddot{X}^\mu + \frac{1}{6} u^3 \ddot{\dot{X}}^\mu \quad (15)$$

and insert it into eq.(11) for the self-force. Following the procedure ¹ analogous to one in [4] we arrive, when the membrane is spherical without oscillations and when its centre of mass speed is much less than the speed of light, at the following result:

$$F_{\text{self}}^r = -\frac{q^2}{2r}\ddot{X}^r + F_{\text{rad}}^r + (\text{higher derivatives}) , \quad r = 1, 2, \dots, D-1 \quad (16)$$

where r is membrane's radius. It has been already found elsewhere [6] that the next term, F_{rad}^r , in eq.(16) is independent of the size of the extended object:

$$F_{\text{rad}}^r = \frac{2}{3}(\ddot{X}^\mu + \dot{X}^\mu \ddot{X}^2) \quad (17)$$

The parameter τ in eqs.(16),(17) is now the proper time of the centroid world line. Since the membrane has finite spatial extension, there is also an infinite series of higher derivative terms in eq.(16).

The total *kinetic momentum* of our spherical membrane can be explicitly calculated from eq.(5). We use $d\sigma_a = n_a d\sigma$, $d\sigma = d\xi^1 d\xi^2$, $n_a \partial^a X^\mu = n^2 \dot{X}^\mu$, $|f| = |\bar{f}|/n^2$, where $\bar{f} \equiv \det f_{ij}$ is the determinant of the induced metric on the spherical surface. Then, after inserting the membrane's constraint $n^2 \dot{X}^2 = 1$, we find

$$P_{\text{m}}^\mu = \kappa \int d\sigma \sqrt{\bar{f}} \frac{\dot{X}^\mu}{\sqrt{\dot{X}^2}} = \kappa 4\pi r^2 \frac{\dot{X}^\mu}{\sqrt{\dot{X}^2}} = m \frac{\dot{X}^\mu}{\sqrt{\dot{X}^2}} \quad (18)$$

where in the last step of eq.(18) we assumed a non-oscillating membrane, $\dot{r} = 0$, so that $\dot{X}^\mu = \dot{X}_C^\mu$ is independent of ξ^1 , ξ^2 . From eq.(18) we read that the membrane's mass is

$$m = 4\pi r^2 \kappa \quad (19)$$

We now insert eqs.(16) and (18) into the equation of motion (5) and identify the coefficient in front of the acceleration as the renormalized or the experimental (observed) mass:

¹In ref.[7] a much simpler procedure is used to calculate the self interaction of a *point particle* with finite electric charge. In the case of our *extended object* such a simple technique cannot be used since an infinitesimal element of the membrane has infinitesimal electric charge; consequently, the integration over the membrane as indicated in eq.(11) has to be performed in order to obtain a final contribution.

$$m_{\text{exp}} = 4\pi r^2 \kappa + \frac{q^2}{2r} \quad (20)$$

The same result (20) for the total membrane's mass - apart from the extra contribution due to radial oscillations) - was obtained in the previous paper [2]. In that work we considered - as a particular case - a spherical charged membrane at rest and identified its mass with the time like component of the *canonical momentum* integrated over the membrane. We also remarked that when world lines or world sheets are oriented into the future, canonical momentum is equal to the momentum obtained from the stress-energy tensor. But in the case of an antiparticle which has its world line or worldsheet oriented into the past, the two kinds of momenta have opposite signs.

4 The 4-momentum of the electromagnetic field around a moving membrane

In the following we are going to calculate the momentum obtained from the stress-energy tensor belonging to our membrane's action (1):

$$T^{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = T_{\text{m}}^{\mu\nu} + T_{\text{EM}}^{\mu\nu} \quad (21)$$

$$T_{\text{m}}^{\mu\nu} = \kappa \int d^d \xi \sqrt{|f|} \partial_a X^\mu \partial^a X^\nu \delta^D(x - X(\xi)) \quad (22)$$

$$T_{\text{EM}}^{\mu\nu} = \frac{1}{16\pi} F_{\rho\sigma} F^{\rho\sigma} g^{\mu\nu} - \frac{1}{4\pi} F^{\rho\mu} F^\nu_\rho \quad (23)$$

where m refers to the "matter" and EM to the electromagnetic field contribution. The momentum is equal to the integral of $T_{\mu\nu}$ over a space-like hypersurface:

$$P^\mu = \int T^{\mu\nu} d\Sigma_\nu = P_{\text{m}}^\mu + P_{\text{EM}}^\mu \quad (24)$$

For the "matter" part we obtain

$$P_m^\mu = \int T_m^{\mu\nu} d\Sigma_\nu = \kappa \int d\sigma^a \sqrt{|f|} \partial_a X^\mu \quad (25)$$

where we have used $d\Sigma_\nu = N_\nu d\Sigma$, $d\Sigma d\tau = d^D x$, and we have decomposed $\partial^a X^\nu = n^a \partial X^\nu + \bar{\partial}^a X^\nu$, so that $\bar{\partial}^a X^\nu N_\nu = 0$, $\partial X^\nu N_\nu = 1$, $d\sigma n^a = d\sigma^a$; N_ν and n_a are normal vector fields to our hypersurface in space-time V_D and world sheet V_d , respectively, and N_ν has been taken parallel to membrane's velocity $\partial X^\nu \equiv \dot{X}^\nu$. Eq.(25) is generally valid and is indeed equal to the expression (5) for the total kinetic momentum. If we now consider a particular case of a two dimensional spherical membrane which is not oscillating radially, then of course we obtain from (25) the expression (18).

The contribution of the electromagnetic field to the momentum is given by

$$P_{EM}^\mu = \int T_{EM}^{\mu\nu} d\Sigma_\nu \quad (26)$$

From the expression (13) we obtain for the electromagnetic field around a generic membrane

$$\begin{aligned} F^{r0} &= \dot{X}^0 \int d\sigma' e(\sigma') \frac{1}{R^3} (X(\xi) - X(\xi'))^r - \dot{X}^r \int d\sigma' e(\sigma') \frac{1}{R^3} (X(\xi) - X(\xi'))^0 \\ &\simeq \int d\sigma' e(\sigma') \frac{1}{R^3} (X(\xi) - X(\xi'))^r \quad \text{for } v \ll c = 1 \end{aligned} \quad (27)$$

$$\begin{aligned} F^{rs} &= \dot{X}^r \int d\sigma' e(\sigma') \frac{1}{R^3} (X(\xi) - X(\xi'))^s - \dot{X}^s \int d\sigma' e(\sigma') \frac{1}{R^3} (X(\xi) - X(\xi'))^r \\ &= \dot{X}^r F^{s0} - \dot{X}^s F^{r0} \end{aligned} \quad (28)$$

For a spherical membrane without oscillations we have

$$F^{r0} = \mathbf{E} = \frac{q}{r^2} \frac{\mathbf{r}}{r} \quad (29)$$

Now we assume as before that

$$d\Sigma_\nu = N_\nu d\Sigma, \quad N_\nu = \dot{X}_\nu \quad (30)$$

so that the line element of the hypersurface over which we integrate is oriented in the same direction as membrane's centre of mass 4-velocity \dot{X}_ν . Then, from eqs.(21-30), we have

$$P^0 = (4\kappa\pi r^2 + \frac{q^2}{2r}) \quad (31)$$

$$\mathbf{P}^r = (4\pi r^2 \kappa + \frac{q^2}{2r}) \mathbf{v} \quad , \quad \mathbf{v} \equiv v^r = \frac{dx^r}{dx^0} \quad , \quad r = 1, 2, 3 \quad (32)$$

where \mathbf{v} is membrane's 3-velocity and $q = \int e(\sigma) d\sigma$ its total electric charge. Our derivation is valid for the speed v much smaller than the speed of light. However, since P^0 is equal to membrane's rest mass, we obtain immediately, by considering a Lorentz transformations, that in general

$$P^\mu = (4\pi\kappa r^2 + \frac{q^2}{2r}) \frac{dx^\mu}{ds} \quad (33)$$

where $ds = dx^0(1 - v^2)^{1/2}$ is an element of the proper time.

In eqs.(31-33) we have again the same result for membrane's mass as calculated before in other ways.

Finally, let us observe that in the literature about the electromagnetic origin of electron mass [6] we find the calculations of the momentum of the electromagnetic field which are analogous to one in eqs.(26-32), except for the choice of the hypersurface element $d\Sigma_\nu$. Instead of orienting $d\Sigma_\nu$ along the electron's velocity, they just consider it oriented into the "laboratory" proper time direction:

$$d\Sigma_\nu = (d\Sigma_0, 0, 0, 0) \quad , \quad d\Sigma_0 = dx^1 dx^2 dx^3 \quad (34)$$

The result for membrane's electromagnetic field energy and momentum is then [6]

$$P_{\text{EM}}^0 = \frac{q^2}{2r} \quad (35)$$

$$\mathbf{P}_{\text{EM}} = \frac{2}{3} \frac{q^2}{r} \mathbf{v} = m_e \mathbf{v} \quad (36)$$

Eqs.(35-36) represent the well known puzzle of the classical electron theory: electron's electromagnetic energy is equal to 3/4 of the mass contained in the electromagnetic field:

$$P_{\text{EM}}^0 = \frac{4}{3} m_{\text{el}} \quad (37)$$

The well known implication, quoted for example by Feynman [6], was that not all the electron's mass is contained in its electromagnetic field. Several ways have been proposed to resolve the problem, including the so called Poincaré stress. However, the most serious difficulty is that the expressions (35,36) are in conflict with Lorentz transformations of P^μ . But our starting expressions, like (26),(13), are all fully Lorentz covariant. Therefore, there must be a mistake in the derivation of energy and momentum as given in (35),(36).

The above old calculations have not taken advantage of the modern fully relativistically covariant formalism, and it has escaped to those authors the obvious fact, that covariance of equation (26) is maintained only if $d\Sigma_\nu$ is chosen consistently [4, 5]; and the result of the integral (26) depends on choice of $d\Sigma_\nu$, since on membrane's surface the divergence relation $T_{\text{EM}}^{\mu\nu},_{\nu} = 0$ is not satisfied [4, 5]. In the literature they inconsistently considered the same hypersurface $d\Sigma_\nu = (d\Sigma_0, 0, 0, 0)$ when calculating electron's electromagnetic *rest* energy P_{rmEM}^0 and when calculating its *momentum* \mathbf{P}_{EM} . However, when particle is at rest, the surrounding magnetic field is zero (see eqs.(28)) and the momentum is zero. If one transforms $P_{\text{EM}}^\mu = (P_{\text{EM}}^0, 0, 0, 0)$ into another Lorentz frame, then also $d\Sigma_\nu = (d\Sigma_0, 0, 0, 0)$, which is pointing along particle's world line, transforms accordingly. In the new frame, $d\Sigma_\nu$ again points along the (moving) particle world line, which is no more the direction of the "laboratory" or the world line of the observer "at rest".

5 Conclusion

We have started from general equations of a moving charged membrane and written down the equations of motion of membrane's centre of mass. We have calculated the contribution of the radiation reaction force to a spherical membrane's mass. Then we have written the general expression for the 4-momentum of the electromagnetic field around a moving membrane. In particular, we calculated the 4-momentum of the field around a spherical membrane. By this two different procedures we obtained the same membrane's mass. Moreover, the resulting membrane's mass coincides with one obtained in [2, 3] where yet another different approach was used. So we have now additional confirmation of our treatment of relativistic charged membranes and their self-interaction. We feel that such a deep understanding of this self-interactive system is interesting, because it might turn out to be a useful step toward a realistic model of electron with finite self-energy, both in its classical and quantized version, including spin [8, 1].

References

- [1] A.O. Barut and M. Pavšič, Mod. Phys. Lett. A7 (1992) 1381
- [2] A.O. Barut and M. Pavšič, Phys. Lett. B306 (1993) 49
- [3] P.A.M. Dirac, Proc. R. Soc.(London) A268 (1962) 57
- [4] A.O. Barut, *Electrodynamics and classical theory of fields and particles* (Macmillan, New York, 1964), p.200
- [5] F. Rohrlich, in *The physicists's conception of nature*, (J.Mehra, Ed., D.Riedel, Dordrecht, 1973), p.331, and references therein
- [6] R.P. Feynman, *Lectures on physics*, (Addison-Wesley, Massachusetts, 1965)
- [7] A.O. Barut and M. Cruz, Phys. Lett. B308 (1993) 247
- [8] A.O. Barut and N. Zanghi, Phys. Rev.Lett. 52 (1984) 2009

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